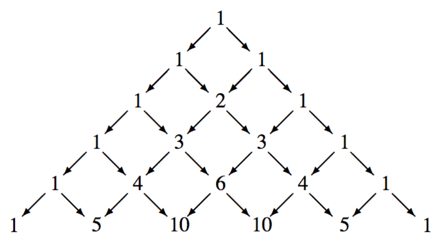
# Hidden Patterns in Pascal's Triangle*

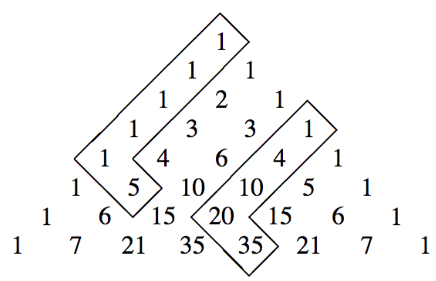
## 1 Warm-Up




1. The figure above is rows 0 through 5 of Pascal's Triangle. Write down the next few rows.

## 2 Patterns in Pascal's Triangle

2. If you add up all the numbers in any row, what do you get? Why does this work?
3. If you alternate the signs of the numbers in any row and then add them together, what do you get? Why does this work?




4. What do you notice about the numbers in the handle and in the tip of these hockey sticks? Why does this work?
5. Can you find any famous sequences of numbers in Pascal's triangle, either staring you in the face or hidden as the sum of some other numbers in Pascal's triangle?
   - Triangular numbers?
   - Tetrahedral numbers?
   - Fibonacci numbers?
   - Other famous numbers?

*From Tom Davis' www.geometer.org

# 3 Coloring Pascal's Triangle

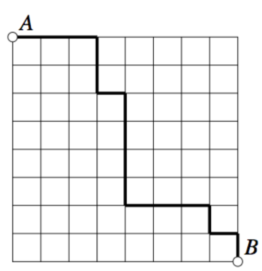
6. Draw a large copy of Pascal's Triangle on graph paper (one square per number), and color each square black if it is an odd number and white if it is an even number. What visual patterns do you find? If you are having trouble fitting large numbers in squares, you could instead write out Pascal's triangle "mod 1" with 1's and 0's, where 1 is for an odd number and 0 is for even.

7. Try the same thing using Pascal's triangle "mod 3": color the squares black if they are not divisible by 3 and white if they are divisible by 3. Or try mod 4, or mod 5, or mod 6.

# 4 Binomial Coefficients

8. Recall that the binomial coefficients, also called the "choose numbers" $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, tell how many ways there are to choose $k$ puppies to take home out of $n$ at the local shelter. How are the numbers in Pascal's Triangle related to the binomial coefficients? Prove that this relationship is true.

9. Suppose we have a grid of city streets, with $m$ north-south streets and $n$ east-west streets. For example, the figure below shows $m = 9$ and $n = 9$, although in general $m$ and $n$ do not need to be the same.




Suppose you start at the northwest tip and walk only south and east. At each intersection, write the number of ways you could get to that intersection. How many ways are there to get to the southeast tip? What does this have to do with Pascal's Triangle?

10. Below are a series of identities involving Pascal's Triangle. Which ones can you prove?

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n} = 2^n \quad \text{(a)}$$

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots+(-1)^n\binom{n}{n} = 0, n>0 \quad \text{(b)}$$

$$\frac{1}{1}\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\cdots+\frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1} \quad \text{(c)}$$

$$0\binom{n}{0}+1\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+n\binom{n}{n} = n2^{n-1} \quad \text{(d)}$$

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\binom{n}{3}^2+\cdots+\binom{n}{n}^2 = \binom{2n}{n} \quad \text{(e)}$$

$$\binom{n}{0}^2-\binom{n}{1}^2+\binom{n}{2}^2-\binom{n}{3}^2+\cdots+(-1)^n\binom{n}{n}^2 = \begin{cases} 0 & : n=2m+1 \\ (-1)^m\binom{2m}{m} & : n=2m \end{cases} \quad \text{(f)}$$

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\binom{n}{6}+\binom{n}{8}+\binom{n}{10}+\cdots = 2^{n-1}, n>0 \quad \text{(g)}$$

$$\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\binom{n}{7}+\binom{n}{9}+\binom{n}{11}+\cdots = 2^{n-1}, n>0 \quad \text{(h)}$$

# 5 Finding formulas for sequences

11. Suppose you come across the sequence of numbers $5, 7, 21, 53, 109, 195, 317, 481, \ldots$ and you would like to find a general formula $f(n)$ such that $f(0) = 5$, $f(1) = 7$, $f(2) = 21$, $f(3) = 53$, and so on. One method that often works is to take differences of successive numbers and write them on the next line, then take differences of these differences, and write them on the next line, etc. See below.

```
5    7    21    53    109    195    317    481    ...
   2    14    32    56     86     122    164    ...
     12    18    24    30     36     42    ...
        6     6     6     6      6     ...
           0     0     0      0     ...
```

How can you use this method to guess the next 3 terms in the original sequence?

12. You can also get an explicit formula from this method. Consider the formula

$$f(n) = 5\binom{n}{0} + 2\binom{n}{1} + 12\binom{n}{2} + 6\binom{n}{3}$$

Evaluate $f(0), f(1), f(2), f(3), f(4), f(5)$ and compare them to your original sequence.

13. Why does this formula work?
14. Use this method to find an explicit formula for $f(n) = 0^3 + 1^3 + 2^3 + 3^3 + \cdots + n^3$.
15. Make up your own sequence of numbers, trade with another student, and see if you can find formula's for each other's sequences.

## 6 Your Turn

16. Look for other patterns in Pascal's Triangle. Try to figure out why these patterns work.