

Knowledge about Knowledge

1 The Forehead Consecutive Number Game

Annie and Zoe have consecutive natural numbers written on their foreheads. This is common knowledge (they both know this, and they both know they know this, etc.) We ask them in turn if they know their own number: “Annie, do you know?” - “Zoe, do you know?” - “Annie, do you know now?”- etc. They only answer “Yes, I do know” or “No, I dont”, and we keep asking until someone figures out their own number (it turns out that the other person then does, too, right on the next question).

1. If Annie has 2 and Zoe has 1, what happens?
2. If Annie has 1 and Zoe has 2, what happens?
3. If Annie has 3 and Zoe has 2, what happens?
4. If Annie has 2 and Zoe has 3, what happens?
5. If Annie has 4 and Zoe has 3, what happens?
6. If Annie has 100 and Zoe has 101, what happens?

2 Red and white hats

The girls in the room have red or white hats. They can see each others but not their own. Every whole hour on the hour, whoever knows her own color stands up and says Got it! (if a girl figures out their own hat color as a result of some other girl standing up, she must keep a poker face and wait for the next round to announce it). All of these rules are common knowledge.

1. Two girls, one red hat, one white hat. Does anything ever happen? (no). A person walks in the room at 12:30 and says Oh, nice to see one-or-more red hats. What happens?
2. Two girls, both have red hats. A person walks in and makes the same statement. What happens? How can that be, given that both girls knew about a red hat long before the statement was made?
3. Three girls, all have red hats. What happens?
4. Ten girls, 3 have red hats, 7 have white hats. What happens?

3 The Forehead Sum Game

Alice, Bob and Charlie have natural numbers x, y, z on their foreheads with $x+y=z$. This is common knowledge. We ask them in turn if they know their own number.

1. Alice=1, Bob=1, Charlie=2: what happens? How about 10, 10, 20? Generally, what happens if two numbers are the same?
2. Alice=1, Bob=2, Charlie=3: what happens? How about 10, 20, 30? Generally, what happens if one number is double another?
3. Alice: No. Bob: No. Charlie: No. Alice: No. Bob: No. Charlie: Yes, its 12. What are the numbers?
 - Were trying to determine the set of numbers on Alices and Bobs foreheads. We dont expect to determine which of them has which number.
 - This is a somewhat harder problem. Think about what youve learned, and what ratio between numbers would yield two rounds of No for Bob.

4 Sum and Product Puzzles

1. Someone picks two integers, X and Y , each from the interval 2 to 99, and tells Mr. Sam the sum of the two integers, and Mr. Paul the product of the two. Sam and Paul do not know the values given to the other. Paul tells Sam, "I do not know the two integers." Sam tells Paul, "I knew you wouldnt. Neither do I." Paul replies, "Oh, now I know the integers." Sam replies back, "Now I know too." Given that both of them are telling the truth, what could the two integers be?
2. Heres another variant, even harder: P and S are given the product and sum of two non-zero digits (1 to 9).
 - (a) P says "I don't know the numbers".
 S says "I don't know the numbers".
 - (b) P says "I don't know the numbers".
 S says "I don't know the numbers".
 - (c) P says "I don't know the numbers".
 S says "I don't know the numbers".
 - (d) P says "I don't know the numbers".
 S says "I don't know the numbers".
 - (e) P says "I know the numbers".