# Dice, Coins, and Card Tricks

## 1 Warm-Up: Ordinary Dice and Coins

1. *Sum of Dice.* When two dice are rolled, there are 36 different outcomes, because $36 = 6 \times 6$. Each outcome is equally likely. Use this table to compute the various sums that can occur. A few cells are already filled in.

| | 1 | 2 | 3 | 4 | 5 | 6 |
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| 1 | 2 | 3 | 4 | | | |
| 2 | | 4 | | | | |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |
| 6 | | | | | | 12 |

   (a) Verify that the probability of rolling two dice and getting a sum of 2 is 1/36

   (b) What is the probability of rolling two dice and getting a sum of 5?

   (c) What is the must likely sum, and what is its probability?

2. *Heads I Win.* Two players alternately toss a penny, and the one that first tosses heads wins. What is the probability that

   (a) the game never ends?

   (b) the first player wins?

   (c) the second player wins?

## 2 Dice

3. *Standard Dice.* What is the probability of rolling six dice and

   (a) getting a sum of 6?

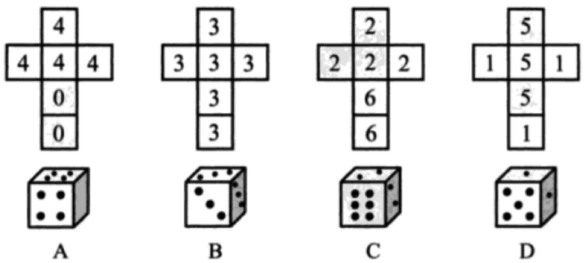
   (b) getting a sum of a 7?

   (c) getting a sum of 10?

   (d) having all six numbers be equal?

   (e) having all six numbers be different?

4. *Non-transitive dice.* Cosider the following four unusual dice.




You pick one die, and then I will pick another. We will each roll our chosen die and the larger number wins. Do you want to play?

You may find it handy to use $6 \times 6$ tables to figure out who wins, when you pair, for example, die A and die B.

| A vs B | 3 | 3 | 3 | 3 | 3 | 3 |
|---|---|---|---|---|---|---|
| 0 | B | | | | | |
| 0 | B | | | | | |
| 4 | A | | | | | |
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5. *Three-way duel.* Alexander Hamilton, Aaron Burr, and Thomas Jefferson fight a 3-cornered pistol duel. All three know that Alexander Hamilton's chance of hitting any target is 1/3, while Aaron Burr *never* misses, and Thomas Jefferson has a 0.5 chance of hitting any target. The way the duel works is that each person is to fire at their choice of target, starting with Alexander Hamilton, and proceding to Aaron Burr, then Thomas Jefferson, then Alexander Hamilton again, etc. (unless someone is hit, in which case they don't shoot), continuing until one person is left unhit. What is Alexander Hamilton's strategy?

   Experiment with different strategies, and simulate the shooting using dice. You can simulate an event with probability 1/3 by tossing one die and seeing if the number shown is 1 or 2, say. Likewise, you can come up with ways to simulate a 1/2 probability event (you don't need a coin, you can still use a die).

   Here are some possible strategies for Alexander Hamilton: shoot at Aaron Burr first; shoot at Thomas Jefferson first; try something else. Choose a strategy, and try simulating the duel. See if you can experimentally estimate the probability that Alexander Hamilton survives, using various strategies.

   Try to justify your experimental conclusions with calculations.

6. *Finite Random Walks.* Suppose you are sitting on the number line, at position $k$, and you flip a fair coin. If it is heads, you move one space to the right. If it is tails, you move one space to the left. You lose if you hit 0 and win if you hit 10. What is the probability that you win?

   Generalize to unfair coins, and let 10 be any positive integer.

7. *Random walk on a cube.* Imagine a bug that crawls along the edges of a cube. The bug does not change directions while traveling on an edge. Two adjacent vertices, $F$ and $P$, have food and poison, respectively. If the bug reaches either of these vertices, it stops traveling. Whenever the bug reaches one of the other six vertices, it has a choice of three edges on which to travel and it chooses randomly (i.e. with probability 1/3 for each choice). For each of these six starting vertices, compute the probability tht the bug lives (i.e. reaches $F$ before reaching $P$).

8. (a) On average, how many times must a die be thrown until a six is rolled?

   (b) How many times, on average, must a die be thrown in order to see all 6 possible outcomes?

## 3 Card Tricks

9. *Face Up Face Down.* I deal cards onto a table, while you tell me whether the card should be face up or face down. You also tell me when to stop dealing the cards. Then you blindfold me. I bet my fingers are so sensitive that I can detect the difference between a face up card and a face down card by touch. I bet that I can divide the cards on the table into two piles, each of which has exactly the same number of face up cards. Do you want to take this bet?

10. *Random Trisection.* A deck of cards is randomly cut into three piles. I bet that at least one of the cards on the top of a pile is a "face card" i.e. a Jack, Queen, or King. Do you want to take this bet?

11. *Random Match* We each have a shuffled deck of cards and we deal our cards one at a time and compare. I bet you that we will have at least one match (for example, our 7th cards are the same). Do you want to take this bet?

12. *Kruskal Count.* A deck of cards is shuffled. You secretly pick a number between 1 and 5. Then you deal out the cards slowly and steadily, face up. Your first key card is the one at the position number that you secretly chose. The value of this card determines how many to deal out to the next key card, e.g., if the key card is a 4, you count off four cards, the last being the new key card. Face cards (Jack, Queen, King) count 5, and Aces count 1. The process is repeated as often as is possible. Eventually you will get a key card (perhaps the last card in the deck) which is not followed by enough cards to get to another one; this last key card is the one you remember. I bet that I can guess your last key card. Do you want to try?

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References: All these problems are from Paul Zeitz, Math Professor, University of San Francisco.

For more on Kruskal Count, see http://www.ams.org/samplings/feature-column/fcarc-mulcahy6.

For more on the Random Match card trick, look up derrangements.

For more on the expected time to see all dice roll outcomes, look up the coupon collector problem.

The non-transitive dice problem is in *Solve This!* by James Tanton.