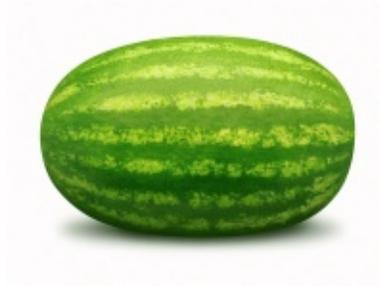


The Watermelon Problem

Warm-up problem: What is the maximum number of pieces that you can cut a watermelon into with 4 straight cuts, if you aren't allowed to rearrange pieces between cuts?



1 Slicing problems in various dimensions

1. What is the maximum number of pieces that you can cut a licorice lace into with n straight cuts? Again, you aren't allowed to rearrange pieces between cuts.
2. What is the maximum number of pieces that you can cut a pizza into with 4 straight cuts? With 5 straight cuts? With n straight cuts? As usual, you aren't allowed to rearrange pieces between cuts.
3. What if you *are* allowed to rearrange pieces between cuts?
4. Fill in as much of the following table as you can.

n # cuts	$L(n)$ max # of pieces of licorice	$P(n)$ max # pieces of pizza	$W(n)$ max # pieces of watermelon
0			
1			
2			
3			
4			
5			
6			

11. Use your observations from parts 9 and 10 to write formulas for $P(n)$ and $W(n)$ in terms of binomial coefficients.

Note: The numbers in Pascal's triangle *are* the binomial coefficients. That is, if we number the rows of Pascal's triangle starting with $n = 0$ for the top row, and number the entries on each row starting with $k = 0$ for the leftmost entry, then the entry in row n and position k is $\binom{n}{k}$, where

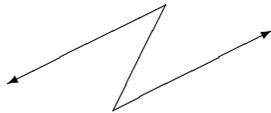
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!}$$

For a proof of this non-obvious fact, see www.mast.queensu.ca/~peter/inprocess/pascal.pdf or www.geometer.org/mathcircles/pascal.pdf.

12. Write down a general formula for the maximum number of pieces you can cut a k -dimensional watermelon into with n straight cuts in terms of binomial coefficients.
13. Use mathematical induction to prove your formulas in part 11. (Hint: use induction to prove the pizza formula first, using your observation from part 6 to justify the inductive step. Then use induction to prove the watermelon formula, using your established formula for $P(n)$ and your observation from part 7 in the inductive step.)
14. Use mathematical induction to prove the general formula in part 12.

3 More slicing and dicing problems

15. What is the minimum number of regions that you can divide space into with 4 distinct planes? The maximum number is the same as the answer to the original watermelon problem. Is it possible to get every number in between the minimum and the maximum? If so, demonstrate; if not, determine which numbers can be achieved and which cannot.
16. Suppose you draw 20 circles in the plane, all passing through the origin, but no two tangent at the origin. Also, except for the origin, no three circles pass through a common point. How many regions are created in the plane? ¹
17. Take a sphere and draw a great circle on it (a great circle is a circle whose center is the center of the sphere). There are two regions created on the surface of the sphere. Now draw another great circle: there are four regions. Now draw a third, not passing through the points of intersection of the first two. How many regions are there? What is the maximum number of regions created by n great circles drawn on the surface of the sphere? ²
18. When you cut the plane with n “zig-zags” (two antiparallel rays joined by a line segment), how many regions can you divide the plane into? ³



19. How many pieces, maximum, can you cut the crescent into with 5 straight cuts? ⁴



^{1,2} From Peter Taylor, www.mast.queensu.ca/~peter/inprocess/cuttingplanes.pdf

³From Joshua Zucker, National Association of Math Circles Problem List: <http://www.mathcircles.org/content/problem-list>

⁴From Sam Lloyd, <http://www.puzzles.com/puzzleplayground/Dissections.htm>