Mathematical Games

For each of these games, two players take turns moving. The winner is the last player who makes a legal move. See if you can find a winning strategy for one of the players.

To describe a winning strategy, you have to explain what the winning player should do in order to win, no matter what the opponent does!

1. **One, Two, Three, Four, Takeaway**
   (a) There are 25 pennies on a table. On each turn, a player can take any number of pennies between 1 and 4. A player loses if he or she unable to move (because there are no pennies left).
   (b) Same game as above but it starts with 24 pennies.
   (c) Same game again, only the initial number of pennies is n.
2. **Lame Tower.** On the top right square of an 8 by 8 chessboard there is a penny that can move either to the left or down through any number of squares. Players take turns moving the penny. A player loses if he or she is unable to move the penny (because it is already on the bottom left square). Consider various other initial positions of the penny.
3. **One, Two, Four Takeaway** There are 25 pennies in a pile. A player can take 1, 2, or 4 pennies on each turn. A player loses if he or she cannot continue (no more pennies left). Try other starting numbers of pennies.
4. **Break the Bar.** You have a rectangular chocolate bar that is 6 x 8 squares in size. At each step, a player takes one piece of the chocolate and breaks it in two along a single straight line bounded by the squares. For example, you can turn the original bar into a 6 x 2 piece and a 6 x 6 piece, and this latter piece can be turned into a 1 x 6 piece and a 5 x 6 piece. The player who cannot make any more breaks loses. What about an $n \times m$ bar?
5. **Two Pile Nim.**
   (a) Now there are two piles of pennies, one pile with 10 pennies and another one with 7. On each turn, a player can take any number of pennies from either one of the two piles. The player unable to move (no pennies left) loses.
   (b) What about if the numbers of pennies in the piles are m and n?

6. **Either, Or, Both.** There are two piles of pennies; one pile contains 10 pennies while the other contains 7. A player can take one penny from the first pile, or one penny from the second pile, or one penny from each of the two piles. The player unable to move loses.

7. **Heads and Tails** Two players take turns placing pennies on a 5×5 checkerboard. The first player puts the pennies down with the heads facing up, and the second player puts tails facing up. At the end of the play, the first player gets a point for each row or column that contains more heads than tails. The second player gets a point for each row or column that contains more tails than heads. The player with the most points wins.

8. **Puppies and Kittens.** There are two piles of pennies; one pile contains 10 and one contains 7. A player can take any number of pennies from the first pile (the puppies), or any number from the second pile (the kittens), or the player can take the same number of pennies from both piles. For example, a player could take 2 from the first pile, or 6 from the second pile, or 3 from each pile. The player unable to move loses.

9. **Three Pile Nim.** There are three piles of pennies; one pile with 6 pennies, a second pile with 5 pennies, and a third pile with 3 pennies. Two players take turns removing any number of pennies from any one of the three piles. The player unable to move loses.

10. **Free a Square.** Two players take turns breaking a piece of chocolate consisting of 5×10 small squares. At each turn, they may break along the division lines of the squares. The player who first obtains a single square of chocolate wins. (What if the first player to free a square loses? This is called the misere version.)

Some of these games are from the book Mathematical Circles: Russian Experience by D. Fomin, S. Genkin, and I. Itenberg.